

Bethe Ansatz in Quantum Mechanics.

2. Construction of Multi-Parameter Spectral Equations¹

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Abstract

In this paper we propose a simple method for building exactly solvable multi-parameter spectral equations which in turn can be used for constructing completely integrable and exactly solvable quantum systems. The method is based on the use of a special functional relation which we call the scalar triangle equation because of its similarity to the classical Yang-Baxter equation.

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1 Introduction

One of the possible ways for building completely integrable quantum systems is the *inverse method of separation of variables* based on the use of the so-called *multi-parameter spectral equations*. The idea is to interpret these equations as the result of separating variables in a certain multi-dimensional completely integrable quantum system and to reconstruct the form of the latter by eliminating some of the spectral parameters considered as separation constants (for details see e.g. refs. [5, 6, 7, 4]).

If one wants to obtain this way exactly solvable completely integrable systems, one should start with *exactly solvable* multi-parameter spectral equations. In this paper we present one possible way of building such equations by considering as an example one-dimensional second-order differential equations. We show that in this case both the multi-parameter spectral equations and their solutions can be constructed from one and the same elementary building blocks called in the paper the ρ -functions which satisfy special functional relations which we called *scalar triangle equations*. The choice of this terminology is not accidental: The reader having some elementary acquaintance with the celebrated r -matrix approach (see e.g. [2]) can easily be convinced that there are many common features between the scalar triangle equations and ordinary matrix triangle equations⁴ There is also a deep relation between the solutions of these equations: the ρ -functions are natural analogues of the classical r -matrices. The general form of the substitution for solving the multi-parameter spectral equations is very similar to that of the famous Bethe Ansatz used for solving completely integrable quantum models. The numerical equations determining the solvability conditions have the same meaning as the corresponding Bethe Ansatz equations.

In this paper we show that there are two essentially different classes of exactly solvable multi-parameter spectral equations which can be constructed from ρ -functions. We call them *rational* and *irrational* equations stressing the fact that in a certain "canonical" coordinate system their "potentials" can be expressed respectively via rational and irrational functions. The rational multi-parameter spectral equations are very well known in the literature. In 1987 Sklyanin obtained such equations [3] as a result of a separation of variables in the so-called completely integrable Gaudin models [1] (which are easily constructable in the framework of the r -matrix approach). Applying to the rational multi-parameter spectral equations the inverse method of separation of variables one recovers the Gaudin models as shown in refs. [5, 6, 7]. As to the irrational multi-parameter spectral equations, these were never discussed in the literature. The reason for this is that the form of the Bethe Ansatz equations determining their spectra drastically differs from the standard one usually obtained in the framework of r -matrix method. Moreover, there is no change of variables which could reduce these equations into standard form. This leads us to the claim that the class of completely integrable quantum systems associated with irrational multi-parameter spectral equations is different from the known classes of models and therefore its investigation is an interesting mathematical problem.

2 The problem

Second-order linear differential equations play an important role in many branches of mathematical physics. By an appropriate homogeneous transformation and a change of variable any

⁴We mean here the so-called classical Yang – Baxter equations.

such equation can be reduced to the following canonical form

$$\left(-\frac{\partial^2}{\partial x^2} + W(x)\right)\psi(x) = 0, \quad (2.1)$$

in which $W(x)$ and $\psi(x)$ are assumed to be analytic functions of the complex variable x .

There are many different mathematical problems which are connected with an equation of the form (2.1). The simplest one can be formulated as follows: for a given function $W(x)$ find the function $\psi(x)$. The general solution of this problem obviously is

$$\psi(x) \sim \sin \varphi \cdot \psi_1(x) + \cos \varphi \cdot \psi_2(x) \quad (2.2)$$

where $\psi_1(x)$ and $\psi_2(x)$ are two linearly independent solutions and φ is an arbitrary mixing angle. The usual way to fix this angle is to impose one additional constraint on the general solution (2.2). This constraint has to be compatible with the linearity of equation (2.1) and thus should have the form

$$\mathcal{L}_0[\psi(x)] = 0, \quad (2.3)$$

where $\mathcal{L}_0[\psi(x)]$ is some appropriately chosen linear functional⁵.

It is well known that, along with this trivial (and mathematically not very interesting) interpretation of equation (2.1), there are many others which lead to richer sets of solutions and are of greater theoretical importance. The essence of most of these interpretations is to allow some freedom in choosing the function $W(x)$ restricting simultaneously the class of allowed functions $\psi(x)$. This leads to the so-called spectral versions of equation (2.1).

Consider an example which will play a central role in our further discussion. Thereby the form of the function $W(x)$ is restricted to

$$W(x) = W_0(x) + \sum_{n=1}^N e_n W_n(x), \quad (2.4)$$

where $W_0(x)$ and $W_1(x), \dots, W_N(x)$ are some fixed functions and e_1, \dots, e_N are arbitrary numerical parameters. Restrict the class of admissible functions $\psi(x)$ by the following $N + 1$ constraints

$$\mathcal{L}_0[\psi(x)] = \mathcal{L}_1[\psi(x)] = \dots = \mathcal{L}_N[\psi(x)] = 0, \quad (2.5)$$

where the $\mathcal{L}_n[\psi(x)]$, $n = 0, \dots, N$ are some arbitrarily chosen linear but linearly independent functionals. Then one can state the problem of finding those values of the parameters e_1, \dots, e_N for which equation (2.1) (with $W(x)$ given by formula (2.4)) has solutions fulfilling equations (2.5).

It is natural to call e_1, \dots, e_N *spectral parameters* and equation (2.1) supplemented by conditions (2.4) and (2.5) a *multi-parameter spectral equation*. The set of admissible values for the parameters e_1, \dots, e_N we shall call the *spectrum*. It is easily seen that the problem (2.1), (2.4), (2.5) is a natural generalization of an ordinary one-parameter spectral equation which corresponds to the case $N = 1$.

It is not difficult to show that in general the spectrum of equations (2.1), (2.4), (2.5) is infinite and discrete. Indeed, let $\psi_1(x, e_1, \dots, e_N)$ and $\psi_2(x, e_1, \dots, e_N)$ denote two linearly independent

⁵In practice, one usually uses the simplest functionals: $\mathcal{L}_0[\psi(x)] \equiv \psi(0)$ and $\mathcal{L}_0[\psi(x)] \equiv \psi'(0)$.

solutions of equation (2.1) considered as functions of the spectral parameters e_1, \dots, e_N . Then the general solution can be written in the form

$$\psi(x) \sim \sin \varphi \cdot \psi_1(x, e_1, \dots, e_N) + \cos \varphi \cdot \psi_2(x, e_1, \dots, e_N) \quad (2.6)$$

where φ is an arbitrary parameter (mixing angle). Substituting (2.6) into (2.5) leads to a system of $N + 1$ numerical equations

$$l_0(\varphi, e_1, \dots, e_N) = l_1(\varphi, e_1, \dots, e_N) = \dots = l_N(\varphi, e_1, \dots, e_N) = 0, \quad (2.7)$$

for $N + 1$ quantities φ and e_1, \dots, e_N . Here $l_n(c, e_1, \dots, e_N)$ denotes the value of the linear functional $\mathcal{L}_n[\psi(x)]$ applied to the solution (2.6). Since the number of equations coincides with the number of unknowns, the spectrum of equation (2.1), (2.4), (2.5) will be discrete in general. Also generally, the function (2.6) is transcendental which suggests that the spectrum should be infinite.

It is however quite obvious that the scheme given cannot be considered a practical way for solving the multi-parameter spectral equations. This is so because for most functions $W_0(x)$ and $W_1(x), \dots, W_N(x)$ the explicit form of the general solution (2.6) is unknown. Thus, one can expect that most of the multi-parameter spectral equations of the form (2.1), (2.4), (2.5) should not be exactly solvable.

Fortunately, there are exceptional cases, and these we intend to discuss in this paper. We think of so-called *exactly solvable* multi-parameter spectral equations which can be solved by means of purely algebraic methods and have many important applications in the theory of completely integrable quantum systems. One of the standard and most effective methods for constructing such equations is the so-called *inverse method*. It rests on a very simple idea: instead of looking for solutions of equation (2.1) for given functions $W_0(x)$ and $W_1(x), \dots, W_N(x)$, one should try to reconstruct the form of these functions starting with appropriately chosen function $\psi(x)$. An advantage of the inverse problem in comparison with the direct one is obvious: the problem of solving differential equations is replaced by the problem of taking derivatives of known functions. However, this is only a relative simplicity: the algebro-analytic part of the inverse problem remains rather non-trivial which is clearly seen from the following discussion:

Let us fix some $K + 1$ functions $\psi_k(x)$, $k = 0, \dots, K$ satisfying constraints (2.5) and $K + 1$ sets of numbers $\{e_{k1}, \dots, e_{kN}\}$, $k = 0, \dots, K$. Substituting these into (2.1), using relation (2.4), taking $e_{k0} = 1$ and dividing finally the k th equation by $\psi_k(x)$, we obtain

$$\sum_{n=0}^N e_{kn} W_n(x) = \psi_k^{-1}(x) \frac{\partial^2 \psi_k(x)}{\partial x^2}, \quad k = 0, 1, \dots, K. \quad (2.8)$$

Formula (2.8) can be considered a system of $K + 1$ linear inhomogeneous equations for $N + 1$ functions $W_0(x)$ and $W_1(x), \dots, W_N(x)$. If $K \leq N$, then system (2.8) is obviously solvable. In this case one easily finds the explicit form of the functions $W_0(x)$ and $W_1(x), \dots, W_N(x)$ for which the equations (2.1), (2.4), (2.5) have $K + 1$ *a priori* known (and, hence, explicit) solutions. The situation changes, however, if $K > N$ (this is just our case, because we are looking for multi-parameter spectral equations having an infinite number of explicit solutions). In this case, the number of equations (2.8) exceeds the number of unknowns which makes the system (2.8) overdetermined for almost all sets of functions $\psi_k(x)$. The only way to get rid of this problem is to start with functions $\psi_k(x)$ for which the compatibility of the equations forming system (2.8) would be guaranteed from the very beginning. But for this to work we

need some reasonable *ansatz* for the functions $\psi(x)$. It is hardly necessary to emphasize that the problem of finding such an *ansatz* is far from being trivial.

From a purely practical point of view, it is much more convenient to deal not with the functions $\psi(x)$ but with their logarithmic derivatives $P(x)$. In terms of the functions $P(x)$, the *ansatz* which we intend to use takes an especially simple form. The substitution

$$\psi(x) = \exp \left\{ \int^x P(x') dx' \right\} \quad (2.9)$$

simplifies also the form of equation (2.1). The new equation

$$W(x) = P^2(x) + P'(x), \quad (2.10)$$

will be considered as starting point for our further considerations.

3 Separable functions of several variables

We shall call a function $f(x_1, \dots, x_k, y_1, \dots, y_l)$ of $k + l$ variables *separable* with respect to the variables x_1, \dots, x_k and y_1, \dots, y_l if it can be represented in the form of a *finite* sum

$$f(x_1, \dots, x_k, y_1, \dots, y_l) = \sum_{n=1}^N g_n(x_1, \dots, x_k) h_n(y_1, \dots, y_l) \quad (3.1)$$

where $h_n(x_1, \dots, x_k)$, $n = 1, \dots, N$ and $g_n(y_1, \dots, y_l)$, $n = 1, \dots, N$ are some functions of k and l variables, respectively. For stressing this property we shall denote such a function by

$$f(x_1, \dots, x_k, y_1, \dots, y_l) = f(x_1, \dots, x_k | y_1, \dots, y_l) \quad (3.2)$$

Functions of several variables separable with respect to all arguments we shall call *completely separable*. Any such function can be represented in the form

$$f(x_1, x_2, \dots, x_m) = \sum_{n=1}^N f_{1,n}(x_1) f_{2,n}(x_2) \dots f_{m,n}(x_m) \quad (3.3)$$

where the $f_{i,n}(x_i)$, $i = 1, \dots, m$ are certain functions of one variable. For such functions we shall use the notation

$$f(x_1, x_2, \dots, x_m) = f(x_1 | x_2 | \dots | x_m) \quad (3.4)$$

Let us now formulate a simple lemma about separable functions.

Lemma 1. Let $r(\xi)$ be a rational function of ξ . Then

$$\frac{r(\xi_1)}{\xi_1 - \xi_2} + \frac{r(\xi_2)}{\xi_2 - \xi_1} = f(\xi_1 | \xi_2) \quad (3.5)$$

and

$$\frac{r(\xi_1)}{(\xi_1 - \xi_2)(\xi_1 - \xi_3)} + \frac{r(\xi_2)}{(\xi_2 - \xi_3)(\xi_2 - \xi_1)} + \frac{r(\xi_3)}{(\xi_3 - \xi_1)(\xi_3 - \xi_2)} = f(\xi_1 | \xi_2 | \xi_3) \quad (3.6)$$

are symmetric and completely separable functions.

Proof. Any rational function can be represented as a linear combination of the so-called *elementary rational functions* having only one singularity in the complex plane. Thus, in order to prove separability of functions (3.5) and (3.6), it is sufficient to make sure that the positions of the singularities of the functions $f(\xi_1|\xi_2)$ and $f(\xi_1|\xi_2|\xi_3)$ with respect to each argument do not depend on the values of other arguments. It is clear that the "dangerous" singularities of both functions (3.5) and (3.6) can only arise from their denominators. However, a simple analysis shows that these singularities (which, obviously, are present in each of the separate terms) cancel each other. The absence of these "dangerous" singularities completes the proof of the lemma.

4 Scalar triangle equation

Let $r(\xi)$ be an arbitrary rational function of a complex variable ξ . Define a complex valued function $\xi(x)$ by the equation

$$(\xi'(x))^2 = r(\xi(x)) \quad (4.1)$$

and construct from it a new function of two complex variables

$$\rho(x, y) = \frac{1}{2} \cdot \frac{\xi'(x) + \xi'(y)}{\xi(x) - \xi(y)}. \quad (4.2)$$

Lemma 2. The function $\rho(x, y)$ obeys the following functional relations:

$$\rho(x, y) + \rho(y, x) = 0, \quad (4.3)$$

$$\rho(x, y)\rho(x, z) + \rho(y, z)\rho(y, x) + \rho(z, x)\rho(z, y) = \omega(x|y|z), \quad (4.4)$$

$$\frac{\partial}{\partial x}\rho(x, y) + \rho^2(x, y) = \omega(x|x|y), \quad (4.5)$$

with $\omega(x|y|z)$ a certain separable function.

Proof. The proof of the anti-symmetry of the function $\rho(x, y)$ immediately follows from definition (4.2). In order to prove the separability of the function $\omega(x|y|z)$ we consider the following chain of equalities

$$\begin{aligned} 4\omega(x|y|z) &= 4[\rho(x, y)\rho(x, z) + \rho(y, z)\rho(y, x) + \rho(z, x)\rho(z, y)] = \\ &= \frac{\xi'(x) + \xi'(y)}{\xi(x) - \xi(y)} \cdot \frac{\xi'(x) + \xi'(z)}{\xi(x) - \xi(z)} + \frac{\xi'(y) + \xi'(z)}{\xi(y) - \xi(z)} \cdot \frac{\xi'(y) + \xi'(x)}{\xi(y) - \xi(x)} + \frac{\xi'(z) + \xi'(x)}{\xi(z) - \xi(x)} \cdot \frac{\xi'(z) + \xi'(y)}{\xi(z) - \xi(y)} = \\ &= \frac{[\xi'(x)]^2}{(\xi(x) - \xi(y))(\xi(x) - \xi(z))} + \frac{[\xi'(y)]^2}{(\xi(y) - \xi(z))(\xi(y) - \xi(x))} + \frac{[\xi'(z)]^2}{(\xi(z) - \xi(x))(\xi(z) - \xi(y))} = \\ &= \frac{r[\xi(x)]}{(\xi(x) - \xi(y))(\xi(x) - \xi(z))} + \frac{r[\xi(y)]}{(\xi(y) - \xi(z))(\xi(y) - \xi(x))} + \frac{r[\xi(z)]}{(\xi(z) - \xi(x))(\xi(z) - \xi(y))}. \end{aligned} \quad (4.6)$$

Lemma 1 then shows that the function $\omega(x|y|z)$ is separable. The last equation (4.5) can be easily proved if we take $z = x + \epsilon$ in (4.4), and take the limit $\epsilon \rightarrow 0$ noting that $\lim_{\epsilon \rightarrow 0} \epsilon \rho(x + \epsilon, x) = 1$. This completes the proof.

Hereafter we shall call equation (4.4) the scalar triangle equation.

5 ξ -functions

Any rational function of $\xi(x)$ and $\xi'(x)$ we shall call a ξ -function. If $F(x)$ is a ξ -function, it can be represented in the form

$$F(x) = R(\xi(x)) + \xi'(x)G(\xi(x)) \quad (5.1)$$

where $R(\xi)$ and $G(\xi)$ are some rational functions. The sum, difference, product and quotient of two ξ -functions is again a ξ -function. The derivative of a ξ -function is again a ξ -function. We call a ξ -function $F(x)$ even if $G(\xi) \equiv 0$ in (5.1) and odd if $R(\xi) \equiv 0$ in (5.1). The product of two odd or two even ξ -functions is an even ξ -function, and the product of an even and an odd ξ -function is an odd ξ -function. This means that the algebra of odd and even ξ -functions is a z_2 -graded algebra. The differential operator $\partial/\partial x$ becomes in this case an odd object.

Lemma 3. If $F(x)$ is a ξ -function, then

$$\sigma(x|y) = [F(x) - F(y)]\rho(x, y) \quad (5.2)$$

is a separable function.

Proof. Substituting (5.1) into (5.2), we can write

$$\sigma(x|y) = \sigma_R(x|y) + \sigma_G(x|y) \quad (5.3)$$

where

$$\sigma_R(x|y) = \frac{R(\xi(x)) - R(\xi(y))}{\xi(x) - \xi(y)} \cdot (\xi'(x) - \xi'(y)) \quad (5.4)$$

and

$$\sigma_G(x|y) = \frac{\xi'(x)G(\xi(x)) - \xi'(y)G(\xi(y))}{\xi(x) - \xi(y)} \cdot (\xi'(x) - \xi'(y)). \quad (5.5)$$

From Lemma 1 it immediately follows that $\sigma_R(x|y)$ is a separable function. In order to prove separability of $\sigma_G(x|y)$, let us rewrite (5.5) in the form

$$\sigma_G(x|y) = \frac{[\xi'(x)]^2 - [\xi'(y)]^2}{\xi(x) - \xi(y)} \cdot (G(\xi(x)) + G(\xi(y))) + \frac{G(\xi(x)) - G(\xi(y))}{\xi(x) - \xi(y)} \cdot (\xi'(x) + \xi'(y))^2. \quad (5.6)$$

Using now (4.1) in the first term and applying then Lemma 1 we find that also $\sigma_G(x|y)$ is separable. This proves the lemma.

6 Bethe ansatz

Let us look for solutions of equation (2.10) in the form

$$P(x) = F(x) + \sum_{i=1}^M \rho(x, x_i) \quad (6.1)$$

where M is an arbitrary non-negative integer, x_1, \dots, x_M are still unknown numbers and $F(x)$ is a ξ -function. We call this form the Bethe Ansatz. Substituting (6.1) into (2.10) gives

$$\begin{aligned} W(x) = & F^2(x) + F'(x) + 2 \sum_{i=1}^M [F(x) - F(x_i)] \rho(x, x_i) + \\ & + 2 \sum_{i=1}^M F(x_i) \rho(x, x_i) + \sum_{i=1}^M \rho^2(x, x_i) + \sum_{i=1}^M \rho'(x, y) + \sum_{i \neq k}^M \rho(x, x_i) \rho(x, x_k). \end{aligned} \quad (6.2)$$

Using formulas (4.4), (4.5) and (5.2), we obtain

$$\begin{aligned} W(x) = & F^2(x) + F'(x) + 2 \sum_{i=1}^M \sigma(x|x_i) + \sum_{i=1}^M \omega(x|x_i) + \sum_{i \neq k}^M \omega(x|x_i|x_k) + \\ & + 2 \sum_{i=1}^M \rho(x, x_i) \left\{ \sum_{k=1, k \neq i}^M \rho(x_i, x_k) + F(x_i) \right\}. \end{aligned} \quad (6.3)$$

We see that the first three terms in the right hand side of (6.3) represent some separable function of x and x_i , while the last sum of the so-called *unwanted terms* is, obviously, non-separable. In order to make the function $W(x)$ separable, one should require all the unwanted terms to vanish. This is equivalent to the system of M equations

$$\sum_{k=1, k \neq i}^M \rho(x_i, x_k) + F(x_i) = 0, \quad i = 1, \dots, M \quad (6.4)$$

which we shall call the Bethe Ansatz equations. If these equations are satisfied then

$$W(x) = F^2(x) + F'(x) + 2 \sum_{i=1}^M \sigma(x|x_i) + \sum_{i=1}^M \omega(x|x_i) + \sum_{i \neq k}^M \omega(x|x_i|x_k) \quad (6.5)$$

and hence can be written in the form

$$W(x) = W_0(x) + \sum_{n=1}^N e_n W_n(x) \quad (6.6)$$

where $W_0(x)$ and $W_n(x)$, $n = 1, \dots, N$ are certain ξ -functions and e_n , $n = 1, \dots, N$ are some numerical coefficients in which all the dependence on numbers x_1, \dots, x_N is contained. We see that equation (2.1) takes in this case the form of a multi-parameter spectral equation. The role of the spectral parameters is played by the numbers e_1, \dots, e_N . The admissible values for these parameters are determined by the solutions of the Bethe Ansatz equations (6.4). It is easily seen that, for any finite M , the number of Bethe Ansatz equations coincides with the number of unknowns and therefore this system has a discrete set of solutions. Since this is an algebraic system, it has a finite number of solutions for any M , but M was an arbitrary non-negative integer. This means that the total number of solutions of system (6.4) is infinite and thus the corresponding multi-parameter spectral equation has infinite discrete and algebraically calculable spectrum.

7 Some simple examples

In this section we consider three simple examples of solutions of the scalar triangle equations and construct the corresponding classes of ξ -functions.

Example 1. Assume that $r(\xi)$ is a first-order polynomial

$$r(\xi) = a + b\xi. \quad (7.1)$$

Then, from (4.1) it follows that

$$\xi'(x) = \sqrt{a + b\xi(x)}. \quad (7.2)$$

Solving this differential equation we obtain

$$\xi(x) = \frac{b(x-t)^2}{4} - \frac{a}{b}, \quad \xi'(x) = \frac{b(x-t)}{2}. \quad (7.3)$$

Construction of the function $\rho(x, y)$ by formula (4.2) gives

$$\rho(x, y) = \frac{1}{x - y}. \quad (7.4)$$

This function obeys all three relations (4.3) – (4.5) with a trivial function $\omega(x|y|z)$:

$$\omega(x|y|z) = 0 \quad (7.5)$$

In this simple case, the set of ξ -functions coincides with the set of all rational functions of x .

Example 2. Let us now assume that $r(\xi)$ is a second-order polynomial:

$$r(\xi) = a + b\xi + c\xi^2. \quad (7.6)$$

Then, from (4.1) it follows that

$$\xi'(x) = \sqrt{a + b\xi(x) + c\xi^2(x)}. \quad (7.7)$$

Solving this differential equation we obtain

$$\xi(x) = \frac{\sqrt{4ac - b^2}}{2c} \sinh \sqrt{c}(x - t) - \frac{b}{2c}, \quad \xi'(x) = \frac{\sqrt{4ac - b^2}}{2\sqrt{c}} \cosh \sqrt{c}(x - t). \quad (7.8)$$

Construction of function $\rho(x, y)$ by formula (4.2) gives in this case

$$\rho(x, y) = \frac{\sqrt{c}}{2} \cdot \operatorname{cth} \frac{\sqrt{c}}{2}(x - y). \quad (7.9)$$

This function obeys all three relations (4.3) – (4.5) with a constant function $\omega(x|y|z)$:

$$\omega(x|y|z) = \frac{c}{4}. \quad (7.10)$$

In this case, the set of ξ -functions coincides with the set of all hyperbolic functions of x with period $2\pi i/\sqrt{c}$. For negative c these functions become trigonometric.

Example 3. Let now $r(x)$ be a third-order polynomial of the form

$$r(\xi) = a + b\xi + c\xi^2 + d\xi^3 \quad (7.11)$$

Then, from (4.1) it follows that

$$\xi'(x) = \sqrt{a + b\xi(x) + c\xi^2(x) + d\xi^3(x)}. \quad (7.12)$$

Solving this differential equation we obtain

$$\xi(x) = \frac{4}{d}\mathcal{P}(x - t, g_2, g_3) - \frac{c}{3d}, \quad \xi'(x) = \frac{4}{d}\mathcal{P}(x - t, g_2, g_3). \quad (7.13)$$

where $\mathcal{P}(x, g_2, g_3)$ denotes the Weierstrass \mathcal{P} -function with

$$g_2 = \frac{c^2 - 3bd}{12}, \quad g_3 = \frac{2c^3 + 9bcd - 27ad^2}{16 \cdot 27} \quad (7.14)$$

Construction of the function $\rho(x, y)$ by formula (4.2) gives

$$\rho(x, y) = \zeta(x - y, g_2, g_3) - \zeta(x - t, g_2, g_3) + \zeta(y - t, g_2, g_3), \quad (7.15)$$

where $\zeta(x, g_2, g_3)$ denotes the Weierstrass ζ -function. This function obeys all three relations (4.3) – (4.5) with a non-trivial function $\omega(x|y|z)$:

$$\omega(x|y|z) = \mathcal{P}(x, g_2, g_3) + \mathcal{P}(y, g_2, g_3) + \mathcal{P}(z, g_2, g_3) \quad (7.16)$$

In this case, the set of ξ -functions coincides with the set of all elliptic functions of x .

8 An equivalent description

Note that equation (2.10) admits an important equivalence transformation which preserves its form and its spectrum. This transformation includes the change of the initial variable x

$$\xi = \xi(x) \quad (8.1)$$

and a linear inhomogeneous transformation of the functions $P(x)$ and $W(x)$:

$$\bar{P}(\xi) = \xi'(x) \left[P(x) + \frac{1}{2} \frac{\partial}{\partial \xi(x)} \ln \xi'(x) \right], \quad (8.2)$$

$$\bar{W}(\xi) = \frac{W(x)}{[\xi'(x)]^2} - \frac{1}{2} \left(\frac{\partial}{\partial \xi(x)} \right)^2 \ln \xi'(x) - \frac{1}{4} \left(\frac{\partial}{\partial \xi(x)} \ln \xi'(x) \right)^2. \quad (8.3)$$

In terms of the new variable ξ and new functions \bar{W} and \bar{P} the equation (2.10) becomes indeed

$$\bar{W}(\xi) = \bar{P}^2(\xi) + \bar{P}'(\xi). \quad (8.4)$$

If we make in (8.4) the substitution

$$\bar{P}(\xi) = \frac{\bar{\psi}'(\xi)}{\bar{\psi}(\xi)} \quad (8.5)$$

then we obtain the transformed version of the initial linear equation (2.1)

$$\left(-\frac{\partial^2}{\partial \xi^2} + \bar{W}(\xi)\right) \bar{\psi}(x) = 0. \quad (8.6)$$

We see that the form of equation (8.6) exactly coincides with that of the initial equation (2.1). Linearity of the transformation (8.3) means that equation (8.6) is again a multi-parameter spectral equation having the same spectrum as (2.1). The equations connected by the transformations (8.2) and (8.3) we shall call equivalent.

From the examples discussed in the previous section we know that the scalar triangle equation has many different solutions which lead to exactly solvable multi-parameter spectral equations expressible in terms of rational, trigonometric, elliptic and also more complicated functions. It is naturally to ask, which of these equations are equivalent in the sense of the transformations (8.2) – (8.3) and which are not. For this one should solve the classification problem. The simplest way to do this is to find some distinguished variable $\xi = \xi(x)$ in terms of which the solutions of the triangle equation have a more or less unified form. The best candidate for a variable is obviously the function $\xi(x)$ obeying equation (4.1). Indeed, in this case all functions $\rho(x, x_i)$ used in the Bethe Ansatz (6.1) transform into functions with the same denominator $\xi - \xi_i$, where $\xi = \xi(x)$ and $\xi_i = \xi(x_i)$. Furthermore, the condition for vanishing of all unwanted (non-separable) terms in expression (6.3) transforms into the condition of regularity of this expression at the points $\xi = \xi_i$. The most natural way to perform the necessary calculations in this case is to start out immediately from the transformed version (8.4) of our equation and use only very general informations on the form of the functions $\bar{W}(\xi)$ and $\bar{P}(\xi)$. This form follows from the results of section 5 and is given as

$$\bar{W}(\xi) = A(\xi) + \sqrt{r(\xi)}B(\xi) \quad (8.7)$$

and

$$\bar{P}(\xi) = a(\xi) + \sqrt{r(\xi)}b(\xi) \quad (8.8)$$

where $A(\xi)$, $B(\xi)$ and $a(\xi)$, $b(\xi)$ are some rational functions and $r(\xi)$ as in (4.1). Substituting (8.7) and (8.8) into (8.4) we obtain two independent equations for $a(\xi)$ and $b(\xi)$

$$A(\xi) = a^2(\xi) + a'(\xi) + r(\xi)b^2(\xi) \quad (8.9)$$

and

$$B(\xi) = b'(\xi) + \left(2a(\xi) + \frac{r'(\xi)}{2r(\xi)}\right)b(\xi) \quad (8.10)$$

In the following two sections we demonstrate that these equations admit two principally different Bethe Ansatz solutions which we call the *rational* and *irrational* ones.

9 The rational Bethe Ansatz solution

Before discussing the general case, consider an important special one. Obviously, the choice

$$b(\xi) = 0 \quad (9.1)$$

leads to a simpler form for system (8.9), (8.10):

$$A(\xi) = a^2(\xi) + a'(\xi), \quad (9.2)$$

and

$$B(\xi) = 0. \quad (9.3)$$

which is hence reduced to only one equation (9.2). The Bethe Ansatz for this equation reads

$$a(\xi) = \alpha(\xi) + \sum_{i=1}^M \frac{1}{\xi - \xi_i} \quad (9.4)$$

Substituting this ansatz into (9.2) and requiring regularity of the function $A(\xi)$ at the points ξ_i , $i = 1, \dots, M$ we obtain

$$A(\xi) = \alpha^2(\xi) + \alpha'(\xi) + 2 \sum_{i=1}^M \frac{\alpha(\xi) - \alpha(\xi_i)}{\xi - \xi_i} \quad (9.5)$$

where the numbers ξ_i , $i = 1, \dots, M$ have to be solutions of the system of Bethe Ansatz equations

$$\sum_{k=1, k \neq i}^M \frac{1}{\xi_i - \xi_k} + \alpha(\xi_i) = 0, \quad i = 1, \dots, M \quad (9.6)$$

Substituting (9.5) and (9.3) into (8.7) and using Lemma 1 we can reduce the function $\bar{W}(\xi)$ to the form

$$\bar{W}(\xi) = \bar{W}_0(\xi) + \sum_{n=1}^N e_n \bar{W}_n(\xi) \quad (9.7)$$

with $\bar{W}_0(\xi)$ and $\bar{W}_n(\xi)$, $n = 1, \dots, N$ certain rational functions and the numbers e_n , $n = 1, \dots, N$ depending on the values of parameters ξ_i , $i = 1, \dots, M$ which fulfill the Bethe Ansatz equations (9.6).

10 The irrational Bethe Ansatz solution

The most general form for the functions $a(x)$ and $b(x)$ for the Bethe Ansatz for $P(x)$ reads

$$a(\xi) = \alpha(\xi) + \sum_{i=1}^M \frac{\alpha_i}{\xi - \xi_i} \quad (10.1)$$

and

$$b(\xi) = \beta(\xi) + \sum_{i=1}^M \frac{\beta_i}{\xi - \xi_i} \quad (10.2)$$

where α_i, β_i, ξ_i , $i = 1, \dots, M$ are some unknown numerical parameters and $\alpha(\xi)$ and $\beta(\xi)$ are fixed rational functions. Substituting formulas (10.1) and (10.2) into equations (8.9) and (8.10)

and requiring regularity of the functions $A(\xi)$ and $B(\xi)$ at the points ξ_i , $i = 1, \dots, M$ we obtain

$$A(\xi) = \alpha^2(\xi) + \alpha'(\xi) + r(\xi)\beta^2(\xi) + \sum_{i=1}^M \frac{1}{(\xi - \xi_i)^2} \left(\frac{r(\xi) - r(\xi_i) - (\xi - \xi_i)r'(\xi_i)}{4r(\xi_i)} \right) + \\ + \sum_{i=1}^M \frac{1}{(\xi - \xi_i)} \left(\alpha(\xi) - \alpha(\xi_i) + \frac{r(\xi)\beta(\xi) - r(\xi_i)\beta(\xi_i)}{\sqrt{r(\xi_i)}} + \frac{r(\xi) - r(\xi_i)}{4} \sum_{i=1}^M \frac{[r(\xi_i)r(\xi_k)]^{-\frac{1}{2}}}{\xi_i - \xi_k} \right) \quad (10.3)$$

and

$$B(\xi) = \beta'(\xi) + \left(2\alpha(\xi) + \frac{r'(\xi)}{4r(\xi)} \right) \beta(\xi) + \\ + \sum_{i=1}^M \frac{1}{\xi - \xi_i} \left(\beta(\xi) - \beta(\xi_i) + \frac{\alpha(\xi) - \alpha(\xi_i)}{\sqrt{r(\xi_i)}} + \frac{r'(\xi)/r(\xi) - r'(\xi_i)/r(\xi_i)}{4\sqrt{r(\xi_i)}} \right) \quad (10.4)$$

where the numbers ξ_i , $i = 1, \dots, M$ are again assumed to be solutions of the system of Bethe Ansatz equations

$$\sum_{k=1, k \neq i}^M \frac{1}{\xi_i - \xi_k} \left(1 + \sqrt{\frac{r(\xi_i)}{r(\xi_k)}} \right) + \sqrt{r(\xi_i)}\beta(\xi_i) + \frac{r'(\xi_i)}{2r(\xi_i)} = 0, \quad i = 1, \dots, M \quad (10.5)$$

For the numbers α_i, β_i , $i = 1, \dots, M$ we obtain

$$\alpha_i = \frac{1}{2}, \quad \beta_i = \frac{1}{2\sqrt{r(\xi_i)}}, \quad i = 1, \dots, M \quad (10.6)$$

Substituting (10.2), (10.3) and (10.4) into (8.7) and using Lemma 1 we can reduce the function $\bar{W}(\xi)$ to the form

$$\bar{W}(\xi) = \bar{W}_0(\xi) + \sum_{n=1}^N e_n \bar{W}_n(\xi) \quad (10.7)$$

where $\bar{W}_0(\xi)$ and $\bar{W}_n(\xi)$, $n = 1, \dots, N$ are certain irrational functions and the numbers e_n , $n = 1, \dots, N$ depend on the parameters ξ_i , $i = 1, \dots, M$ which satisfy the Bethe Ansatz equations (10.5).

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